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A GENERALIZED CLASS OF JACK-KNIFED ESTIMATOR FOR PRODUCT OF POPULATION MEANS USING TWO AUXILIARY VARIABLES UNDER MEASUREMENT ERRORS

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Abstract— *In this paper, we have suggested a generalized estimator for the estimation of product of population means in the presence of measurement errors. The expression of bias and mean square error are derived under measurement errors model. Further, we have also proposed a jack-knife version of the estimators based on Quenouille's method in presence of measurement errors. Finally, some concluding remarks are made clearly demonstrating the importance of the proposed classes.*

Keywords— *Measurement error, Jack-knife estimator, Unbiasedness, Mean square error.*

I. INTRODUCTION

In sampling from finite population, the estimators based on data explicitly assume that the observations are correct measurements on characteristics of study. But unfortunately this ideal situation is not met in practice for a varied of reasons, such as non-response errors, reporting errors and computational errors etc. when the measurement errors are negligible, the statistical inferences based on observed data continues to remain valid. On the contrary when they are not appreciably small or negligible, the inferences may simply be invalid or inaccurate but may also often lead to unexpected, undesirable and unfortunate consequences.

Over the past several decades, statisticians are paying their attention towards the problem of estimation of parameters in the presence of measurement errors. In survey sampling, the properties of data usually presuppose that the observations are the correct measurements on characteristics being studied. However this assumption is not satisfied in many applications and data is contaminated with measurement errors, such as non-response errors, reporting errors and computing errors. These measurement errors make the result invalid, which are meant for no measurement error case. If measurement errors are very small and we can neglect it, then the statistical inference based on data observed continue to remain valid. On the contrary, when they are not appreciably small and negligible, the inferences may not be simply invalid and inaccurate but may often lead to unexpected, undesirable and unfortunate consequences (see Srivastava and Shalabh, 2001). Some important sources of measurement errors in survey data are discussed in Cochran (1968), Shalabh (1997), and Singh and Karpe (2008, 2010) studied some estimators of population mean under measurement errors.

For a simple random sample of size n, let $(y_{1j}, y_{2j}, x_{1j}, x_{2j})$ be the observed values instead of the true values $(Y_{1j}, Y_{2j}, X_{1j}, X_{2j})$ on the characteristics (Y_1, Y_2, X_1, X_2) respectively. Let the measurement error be defined as

$$u_{1,i} = y_{1,i} - Y_{1,i} \quad (1.1)$$

$$u_{2,i} = y_{2,i} - Y_{1,i} \quad (1.2)$$

$$v_{1,i} = x_{1,i} - X_{1,i} \quad (1.3)$$

$$v_{2,i} = x_{2,i} - X_{2,i} \quad (1.4)$$

here i=1, 2, …n

which are stochastic in nature and are uncorrelated with mean zero and variances $\sigma^2_{u_1}$, $\sigma^2_{u_2}$, $\sigma^2_{v_1}$ and $\sigma^2_{v_1}$ respectively. Further, let population means of (Y_1, Y_2, X_1, X_2) be $(\mu_{X_1}, \mu_{Y_1}, \mu_{X_2}, \mu_{Y_2})$ and the population variance of (Y_1, Y_2, X_1, X_2) be $(\sigma^2_{X_1}, \sigma^2_{Y_1}, \sigma^2_{X_2}, \sigma^2_{Y_2})$ respectively. $\sigma_{X_1Y_1}, \sigma_{X_2Y_1}, \sigma_{X_1Y_2}$, $\sigma_{X_2Y_2}, \sigma_{X_1X_2}$ and $\sigma_{Y_1Y_2}$ be the population correlation coefficient between $(X_1, Y_1), (X_2, Y_1), (X_1, Y_2), (X_2, Y_2), (X_1, X_2)$ and (Y_1, Y_2) respectively.

2 PROPOSED CLASS OF GENERALIZED ESTIMATOR

The generalized estimator of product of population mean $\hat{P}_h$ using estimated product of means (Y_1, Y_2) be $\hat{P} = \hat{\mu}_{Y_1}\hat{\mu}_{Y_2}$ and μ_1 and μ_2 of auxiliary variable X_1 and X_2 respectively is

$$\hat{P}_h = h\left[\hat{P}, \frac{\bar{x}_1}{\mu_1}, \frac{\bar{x}_2}{\mu_2}\right] \quad (2.1)$$

where $(\bar{x}_1, \bar{x}_2)$ are the sample means of X_1, X_2 and $\hat{P} = y_1 y_2$ is the estimate of $P = \mu_{Y_1}\mu_{Y2}$ respectively for a simple random variable of size n. Also, $h(.)$ is the function of $\overline{y_1 y_2}$ $\frac{\bar{x}_1}{\mu_1}$ and $\frac{\bar{x}_2}{\mu_2}$ satisfying (i). $h(P,1,1)=P$ (2.2)

(ii).The function is continuous and bounded in the closed interval of real line **R.**

The first and second order partial derivatives of the function are exist and are continuous and bounded in **R**.

Consider the following terms

$$\bar{y}_1 = \frac{1}{n}\left[\sum_{j=1}^{n} Y_{1,j}\right]$$

$$= \frac{1}{n}\left[\sum_{j=1}^{n} u_{1,j} + Y_{1,j} - \mu_{Y_1} + \mu_{Y_1}\right]$$

$$= \frac{1}{\sqrt{n}}\left[W_{u_1} + W_{Y_1}\right] + \mu_{Y_1} \quad (2.3)$$

Similarly, we get

$$\bar{y}_2 = \frac{1}{\sqrt{n}}\left[W_{u_2} + W_{Y_2}\right] + \mu_{Y_2} \quad (2.4)$$

$$\bar{x}_1 = \frac{1}{\sqrt{n}}\left[W_{v_1} + W_{x_1}\right] + \mu_1 \quad (2.5)$$

and

$$\bar{x}_2 = \frac{1}{\sqrt{n}}\left[W_{v_2} + W_{x_2}\right] + \mu_2 \quad (2.6)$$

On expand $\hat{P}$ using Maclaurin series expansion and ignore the terms of having the powers greater than two, we have

$$\hat{P}=P\left(1+\frac{W_{u_1}+W_{y_1}}{n^{1/2}\mu_{Y_1}}\right)\left(1+\frac{W_{u_2}+W_{y_2}}{n^{1/2}\mu_{Y_2}}\right)$$

$$=P\left(1+\frac{W_{u_1}+W_{y_1}}{n^{1/2}\mu_{Y_1}}+\frac{W_{u_2}+W_{y_2}}{n^{1/2}\mu_{Y_2}}+\frac{\left(W_{u_1}+W_{y_1}\right)\left(W_{u_2}+W_{y_2}\right)}{n^{1/2}\mu_{Y_1}\, n^{1/2}\mu_{Y_2}}\right) \quad (2.7)$$

3 BIAS AND MSE OF THE PROPOSED CLASS OF GENERALIZED ESTIMATOR

To obtain the bias of the generalized class of estimator we expand (2.1) by Taylor's series expansion up to third order terms about the point $(P,1,1)$ and noting that $h(P,1,1)=P$ and is continuous and bounded in the closed interval of real line **R**. We assume that the first and second order partial derivatives of the function $h(.)$ are exist and are continuous and bounded in **R**.

$$\hat{P} - P=\left[P\left(\frac{W_{u_1}+W_{y_1}}{n^{1/2}\mu_{Y_1}}+\frac{W_{u_2}+W_{y_2}}{n^{1/2}\mu_{Y_2}}+\frac{\left(W_{u_1}+W_{y_1}\right)\left(W_{u_2}+W_{y_2}\right)}{n^{1/2}\mu_{Y_1}\, n^{1/2}\mu_{Y_2}}\right)h_1\right.$$

$$+\frac{W_{v_1}+W_{x_1}}{n^{1/2}\mu_1}h_2+\frac{W_{v_2}+W_{x_2}}{n^{1/2}\mu_2}h_3+\frac{1}{2!n}\left\{P^2\left(\frac{W_{u_1}+W_{y_1}}{\mu_{Y_1}}+\frac{W_{u_2}+W_{y_2}}{\mu_{Y_2}}\right)^2 h_{300}\right.$$

$$+\frac{\left(W_{v_1}+W_{x_1}\right)^2}{\mu_1^2}h_{030}+\frac{\left(W_{v_2}+W_{x_2}\right)^2}{\mu_2^2}h_{003}+2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W_{v_1}+W_{x_1}\right)}{\mu_{Y_1}\mu_1}h_{110}$$

$$+2\frac{P\left(W_{u_2}+W_{y_2}\right)\left(W_{v_1}+W_{x_1}\right)}{\mu_{Y_2}\mu_1}h_{110}+2\frac{\left(W_{v_1}+W_{x_1}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_1\mu_2}h_{011}$$

$$\left.\left.+2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_{Y_2}\mu_1}h_{101}+2\frac{P\left(W_{u_2}+W_{y_2}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_{Y_2}\mu_2}h_{101}\right\}\right] \quad (3.1)$$

On taking expectation on both side of (3.1), the bias of $\hat{R}_g$ is

$$\text{Bias}\left(\hat{P}_g\right)=\frac{P}{n}\left[\left(\frac{1}{2\mu_1^2}\sigma^2_{X_1}h_{030}+\frac{1}{2\mu_2^2}\sigma^2_{X_2}g_{003}+\frac{1}{P\mu_1\mu_2}\sigma_{X_1X_2}h_{011}\right)\right.$$

$$\left.+\frac{1}{\mu_{Y_1}}\left(\frac{1}{\mu_1}\sigma_{X_1Y_1}h_{110}+\frac{1}{\mu_2}\sigma_{X_2Y_1}h_{101}\right)+\frac{1}{\mu_{Y_2}}\left(\frac{1}{\mu_1}\sigma_{X_1Y_2}h_{110}+\frac{1}{\mu_2}\sigma_{X_2Y_2}h_{101}\right)\right]$$

$$+\frac{1}{n}\left[\frac{1}{2\mu_1^2}\sigma^2_{v_1}h_{030}+\frac{1}{2\mu_2^2}\sigma^2_{v_2}h_{003}\right] \quad (3.2)$$

Now squairing the both sides of (3.1) and taking expectation, we get mean Square Error of $\hat{P}_g$ given by

$$MSE\left(\hat{P}_g\right)=E\left(\hat{P}_g-P\right)^2$$

$$=\left[P\left(\frac{W_{u_1}+W_{y_1}}{n^{1/2}\mu_{Y_1}}\right)h_1+\frac{W_{v_1}+W_{x_1}}{n^{1/2}\mu_1}h_2+\frac{W_{v_2}+W_{x_2}}{n^{1/2}\mu_2}h_3\right]^2$$

$$=\frac{P^2}{n}\left[\left(\frac{1}{\mu^2_{Y_1}}\sigma^2_{Y_1}+\frac{1}{\mu^2_{Y_2}}\sigma^2_{Y_2}+\frac{2}{\mu_{Y_1}\mu_{Y_2}}\sigma_{Y_1Y_2}\right)h_1^2\right.$$

$$+\left(\frac{1}{P^2\mu_1^2}\sigma^2_{X_1}h_2^2+\frac{1}{P^2\mu_2^2}\sigma^2_{X_2}h_3^2+\frac{2}{P^2\mu_1\mu_2}\sigma_{X_1X_2}h_2h_3\right)$$

$$\left.+\frac{2}{\mu_{Y_1}}\left(\frac{1}{\mu_1}\sigma_{X_1Y_1}h_1h_2+\frac{1}{\mu_2}\sigma_{X_2Y_1}h_1h_3\right)+\frac{2}{\mu_{Y_2}}\left(\frac{1}{\mu_1}\sigma_{X_1Y_2}h_1h_2+\frac{1}{\mu_2}\sigma_{X_2Y_2}h_1h_3\right)\right]$$

$$+\frac{P^2}{n}\left[\left(\frac{1}{\mu^2_{Y_2}}\sigma^2_{u_1}+\frac{1}{\mu^2_{Y_2}}\sigma^2_{u_2}\right)h_1^2+\left(\frac{1}{P^2\mu_1^2}\sigma^2_{V_1}h_2^2+\frac{1}{P^2\mu_2^2}\sigma^2_{V_2}h_3^2\right)\right] \quad (3.3)$$

4 PROPOSED CLASS OF GENERALIZED UNBIASED ESTIMATOR

We consider a simple random sample of size n=2m and split this sample randomly into two sub samples each of size m. Let $\left(\bar{y}_{1,2m}\bar{y}_{2,2m}\right)$ be the product of sample means of values on Y_1Y_2 and $\left(\underline{x}_{1,2m},\underline{x}_{2,2m}\right)$ be the sample means of $\left(X_1,X_2\right)$ respectively for the entire sample of size 2m and $\bar{y}^{(i)}_{1,m}y^{(i)}_{2,m}$ be the product of sample means of Y_1Y_2 and $\left(\bar{x}_{1,2m},\bar{x}_{2,2m}\right)$ be the sample means of $\left(X_1,X_2\right)$ respectively for i_{th} $(i=1,2)$ sub-sample of size m.

The generalized estimator based on sample of size 2m of product of population means $\mu_{Y_1}\mu_{Y_2}$ using means μ_1 and μ_2 of auxiliary variables X_1 and X_2, $\hat P_H^{(1)}$ and $\hat P_H^{(2)}$ be the generalized estimators for the two randomly split sub-samples of size m each, respectively, given by

$$\hat P_H^{(3)} = H\left(\hat P_{2m}, \frac{\bar x_{1,2m}}{\mu_1}, \frac{\bar x_{2,2m}}{\mu_2}\right) \quad (4.1)$$

and

$$\hat P_H^{(1)} = H\left(\hat P_m^{(1)}, \frac{\bar x^{(1)}_{1,m}}{\mu_1}, \frac{\bar x^{(1)}_{2,m}}{\mu_2}\right) \quad (4.2)$$

$$\hat P_H^{(2)} = H\left(\hat P_m^{(2)}, \frac{\bar x^{(2)}_{1,m}}{\mu_1}, \frac{\bar x^{(2)}_{2,m}}{\mu_2}\right) \quad (4.3)$$

where

$H\left[\hat P_{2m}, \frac{\bar x_{1,2m}}{\mu_1}, \frac{\bar x_{2,2m}}{\mu_2}\right]$, $H\left[\hat P_m^{(1)}, \frac{\bar x^{(1)}_{1,m}}{\mu_1}, \frac{\bar x^{(1)}_{2,m}}{\mu_2}\right]$ and $H\left[\hat P_m^{(2)}, \frac{\bar x^{(2)}_{1,m}}{\mu_1}, \frac{\bar x^{(2)}_{2,m}}{\mu_2}\right]$ are the bounded functions of $\left(\hat P_{2m}, \frac{\bar x_{1,2m}}{\mu_1}, \frac{\bar x_{2,2m}}{\mu_2}\right)$, $\left(\hat P_m^{(1)}, \frac{\bar x^{(1)}_{1,m}}{\mu_1}, \frac{\bar x^{(1)}_{2,m}}{\mu_2}\right)$ and $\left(\hat P_m^{(2)}, \frac{\bar x^{(2)}_{1,m}}{\mu_1}, \frac{\bar x^{(2)}_{2,m}}{\mu_2}\right)$ respectively satisfying the regularity conditions of $h(.)$ in (2.1) for $H(.)$ involved in generalized estimator $\hat P_H^{(i)}$ where i=1, 2, 3.

On the lines of Sukhatme and Sukhatme (Chapter IV, page 162), for N to be large we consider the Jack-knife generalized estimator $\hat P_{Hj}$ given by

$$\hat P_{Hj} = 2\hat P_H^{(3)} - \frac{1}{2}\left(\hat P_H^{(1)} + \hat P_H^{(2)}\right) \quad (4.4)$$

Now, consider

$$\hat P_{2m} = \sum_1^{2m} y_{1,2m} \sum_1^{2m} y_{2,2m}$$

$$= P\left(1 + \frac{W_{u_1}+W_{y_1}}{(2m)^{1/2}\mu_{Y_1}} + \frac{W_{u_2}+W_{y_2}}{(2m)^{1/2}\mu_{Y_2}} + \frac{\left(W_{u_1}+W_{y_1}\right)\left(W_{u_2}+W_{y_2}\right)}{(2m)^{1/2}\mu_{Y_1}\,(2m)^{1/2}\mu_{Y_2}}\right) \quad (4.5)$$

and $\sum_m \sum_{1,m} \sum_{2,m}$

$$\hat P^{(i)} = \sum_1 y^{(i)} \sum_1 y^{(i)}$$

$$= P\left(1 + \frac{W^{(i)}_{u_1}+W^{(i)}_{y_1}}{(m)^{1/2}\mu_{Y_1}} + \frac{W^{(i)}_{u_2}+W^{(i)}_{y_2}}{(m)^{1/2}\mu_{Y_2}} + \frac{\left(W^{(i)}_{u_1}+W^{(i)}_{y_1}\right)\left(W^{(i)}_{u_2}+W^{(i)}_{y_2}\right)}{(m)^{1/2}\mu_{Y_1}\,(m)^{1/2}\mu_{Y_2}}\right) \quad (4.6)$$

where i=1,2

Also,

$$\bar x_{1,2m} = \frac{1}{(2m)^{1/2}}\left[W_{v_1} + W_{x_1}\right] + \mu_1 \quad (4.7)$$

$$\bar x_{2,2m} = \frac{1}{(2m)^{1/2}}\left[W_{v_2} + W_{x_2}\right] + \mu_2 \quad (4.8)$$

$$\bar x^{(1)}_{1,m} = \frac{1}{(m)^{1/2}}\left[W^{(1)}_{v_1} + W^{(1)}_{x_1}\right] + \mu_1 \quad (4.9)$$

$$\bar x^{(1)}_{2,m} = \frac{1}{(m)^{1/2}}\left[W^{(1)}_{v_2} + W^{(1)}_{x_2}\right] + \mu_2 \quad (4.10)$$

$$\bar x^{(2)}_{1,m} = \frac{1}{(m)^{1/2}}\left[W^{(2)}_{v_1} + W^{(2)}_{x_1}\right] + \mu_1 \quad (4.11)$$

$$\bar x^{(2)}_{2,m} = \frac{1}{(m)^{1/2}}\left[W^{(2)}_{v_2} + W^{(2)}_{x_2}\right] + \mu_2 \quad (4.12)$$

We expand (4.1), (4.2) and (4.3) similarly as (3.1) using Taylor's series expansion up to first order of approximation, to get

$$\hat P_H^{(3)} - P = \Bigg[P\left(\frac{W_{u_1}+W_{y_1}}{(2m)^{1/2}\mu_{Y_1}} + \frac{W_{u_2}+W_{y_2}}{(2m)^{1/2}\mu_{Y_2}} + \frac{\left(W_{u_1}+W_{y_1}\right)}{(2m)^{1/2}\mu_{Y_1}}\frac{\left(W_{u_2}+W_{y_2}\right)}{(2m)^{1/2}\mu_{Y_2}}\right)H_1 + \frac{W_{v_1}+W_{x_1}}{(2m)^{1/2}\mu_1}H_2 - \frac{W_{v_2}+W_{x_2}}{(2m)^{1/2}\mu_2}H_3 + \frac{1}{2!(2m)}\Bigg\{P^2\left(\frac{W_{u_1}+W_{y_1}}{\mu_{Y_1}} - \frac{W_{u_2}+W_{y_2}}{\mu_{Y_2}}\right)^2 H_{300} + \frac{\left(W_{v_1}+W_{x_1}\right)^2}{\mu_1^2}H_{030} + \frac{\left(W_{v_2}+W_{x_2}\right)^2}{\mu_2^2}H_{003} + 2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W_{v_1}+W_{x_1}\right)}{\mu_{Y_1}\mu_1}H_{110} + 2\frac{P\left(W_{u_2}+W_{y_2}\right)\left(W_{v_1}+W_{x_1}\right)}{\mu_{Y_2}\mu_1}H_{110} + 2\frac{\left(W_{v_1}+W_{x_1}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_1\mu_2}H_{011} + 2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_{Y_1}\mu_2}H_{101} + 2\frac{P\left(W_{u_2}+W_{y_2}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_{Y_2}\mu_2}H_{101}\Bigg\}\Bigg] \quad (4.13)$$

For i=1,2, we have

$$\hat P^{(i)} - P = \Bigg[P\left(\frac{W^{(i)}_{u_1}+W^{(i)}_{y_1}}{m^{1/2}\mu_{Y_1}} - \frac{W^{(i)}_{u_2}+W^{(i)}_{y_2}}{m^{1/2}\mu_{Y_2}} + \frac{\left(W^{(i)}_{u_1}+W^{(i)}_{y_1}\right)\left(W^{(i)}_{u_2}+W^{(i)}_{y_2}\right)}{m^{1/2}\mu_{Y_1}\,m^{1/2}\mu_{Y_2}}\right)H_1 + \frac{W^{(i)}_{v_1}+W^{(i)}_{x_1}}{m^{1/2}\mu_1}H_2 + \frac{W^{(i)}_{v_2}+W^{(i)}_{x_2}}{m^{1/2}\mu_2}H_3 + \frac{1}{2!m}\Bigg\{P^2\left(\frac{W^{(i)}_{u_1}+W^{(i)}_{y_1}}{\mu_{Y_1}} + \frac{W^{(i)}_{u_2}+W^{(i)}_{y_2}}{\mu_{Y_2}}\right)^2 H_{300} + \frac{\left(W^{(i)}_{v_1}+W^{(i)}_{x_1}\right)^2}{\mu_1^2}H_{030} + \frac{\left(W^{(i)}_{v_2}+W^{(i)}_{x_2}\right)^2}{\mu_2^2}H_{003} + 2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W^{(i)}_{v_1}+W^{(i)}_{x_1}\right)}{\mu_{Y_1}\mu_1}H_{110}$$

$$+2\frac{P\left(W_{u_2}^{(i)}+W_{y_2}^{(i)}\right)\left(W_{v_1}^{(i)}+W_{x_1}^{(i)}\right)}{\mu_{Y_2}\mu_1}H_{110}+2\frac{\left(W_{v_1}^{(i)}+W_{x_1}^{(i)}\right)\left(W_{v_2}^{(i)}+W_{x_2}^{(i)}\right)}{\mu_1\mu_2}H_{011}$$

$$+2\frac{P\left(W_{u_1}^{(i)}+W_{y_1}^{(i)}\right)\left(W_{v_2}^{(i)}+W_{x_2}^{(i)}\right)}{\mu_{Y_1}\mu_1}H_{101}+2\frac{P\left(W_{u_2}^{(i)}+W_{y_2}^{(i)}\right)\left(W_{v_2}^{(i)}+W_{x_2}^{(i)}\right)}{\mu_{Y_2}\mu_2}P_{101}\Bigg\}\Bigg] \quad (4.14)$$

Substituting these values in (4.4), we have

$$\hat{P}_{Hj}-P=2\Bigg[P\left\{\frac{W_{u_1}+W_{y_1}}{(2m)^{1/2}\mu_{Y_1}}+\frac{W_{u_2}+W_{y_2}}{(2m)^{1/2}\mu_{Y_2}}+\frac{\left(W_{u_1}+W_{y_1}\right)\left(W_{u_2}+W_{y_2}\right)}{(2m)^{1/2}\mu_{Y_1}(2m)^{1/2}\mu_{Y_2}}\right\}H_1$$

$$+\frac{W_{v_1}+W_{x_1}}{(2m)^{1/2}\mu_1}H_2+\frac{W_{v_2}+W_{x_2}}{(2m)^{1/2}\mu_2}H_3$$

$$+\frac{1}{2!(2m)}\Bigg\{P^2\left(\frac{W_{u_1}+W_{y_1}}{\mu_{Y_1}}+\frac{W_{u_2}+W_{y_2}}{\mu_{Y_2}}\right)^2H_{300}$$

$$+\frac{\left(W_{v_1}+W_{x_1}\right)^2}{\mu_1^2}H_{030}+\frac{\left(W_{v_2}+W_{x_2}\right)^2}{\mu_2^2}H_{003}$$

$$+2\frac{\left(W_{v_1}+W_{x_1}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_1\mu_2}H_{011}$$

$$+2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W_{v_1}+W_{x_1}\right)}{\mu_{Y_1}\mu_1}H_{110}$$

$$+2\frac{H\left(W_{u_2}+W_{y_2}\right)\left(W_{v_1}+W_{x_1}\right)}{\mu_{Y_2}\mu_1}H_{110}$$

$$+2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_{Y_2}\mu_1}H_{101}$$

$$+2\frac{P\left(W_{u_2}+W_{y_2}\right)\left(W_{v_2}+W_{x_2}\right)}{\mu_{Y_2}\mu_2}H_{101}\Bigg\}\Bigg]$$

$$-\frac{1}{2}\sum_{i=1}^{2}\Bigg[P\left(\frac{W_{u_1}^{(i)}+W_{y_1}^{(i)}}{m^{1/2}\mu_{Y_1}}+\frac{W_{u_2}^{(i)}+W_{y_2}^{(i)}}{m^{1/2}\mu_{Y_2}}+\frac{\left(W_{u_1}^{(i)}+W_{y_1}^{(i)}\right)\left(W_{u_2}^{(i)}+W_{y_2}^{(i)}\right)}{m^{1/2}\mu_{Y_1}\,m^{1/2}\mu_{Y_2}}\right)H_1$$

$$+\frac{W_{v_1}^{(i)}+W_{x_1}^{(i)}}{m^{1/2}\mu_1}H_2+\frac{W_{v_2}^{(i)}+W_{x_2}^{(i)}}{m^{1/2}\mu_2}H_3$$

$$+\frac{1}{2!m}\left\{P^2\left(\frac{W_{u_1}^{(i)}+W_{y_1}^{(i)}}{\mu_{Y_1}}+\frac{W_{u_2}^{(i)}+W_{y_2}^{(i)}}{\mu_{Y^2}}\right)^2H_{300}\right.$$

$$+\frac{\left(W_{v_1}^{(i)}+W_{x_1}^{(i)}\right)^2}{\mu_1^2}H_{030}+\frac{\left(W_{v_2}^{(i)}+W_{x_2}^{(i)}\right)^2}{\mu_2^2}H_{003}$$

$$+2\frac{P\left(W_{u_1}+W_{y_1}\right)\left(W_{v_1}^{(i)}+W_{x_1}^{(i)}\right)}{\mu_{Y_1}\mu_1}H_{110}$$

$$+2\frac{P\left(W_{u_2}^{(i)}+W_{y_2}^{(i)}\right)\left(W_{v_1}^{(i)}+W_{x_1}^{(i)}\right)}{\mu_{Y_2}\mu_1}H_{110}$$

$$+2\frac{\left(W_{v_1}^{(i)}+W_{x_1}^{(i)}\right)\left(W_{v_2}^{(i)}+W_{x_2}^{(i)}\right)}{\mu_1\mu_2}H_{011}$$

$$+2\frac{P\left(W_{u_1}^{(i)}+W_{y_1}^{(i)}\right)\left(W_{v_2}^{(i)}+W_{x_2}^{(i)}\right)}{\mu_{Y_1}\mu_1}H_{101}$$

$$+2\frac{P\left(W_{u_2}^{(i)}+W_{y_2}^{(i)}\right)\left(W_{v_2}^{(i)}+W_{x_2}^{(i)}\right)}{\mu_{Y_2}\mu_2}H_{101}\Bigg\}\Bigg] \quad (4.15)$$

On taking expectation on both sides (4.15) the bias of the $\hat{P}_{Hj}$ up to terms of order $O\left(\frac{1}{m}\right)$, is given by

$$\text{Bias}\left(\hat{P}_{Hj}\right)=0 \quad (4.16)$$

On squaring both side of (4.15) and taking expectation, the mean square error of $\hat{R}_{Gj}$ up to terms of order $O\left(\frac{1}{m}\right)$, is given by

$$MSE\left(\hat{P}_{Hj}\right)=E\left(\hat{P}_{Hj}-P\right)^2$$

$$=E\Bigg[2P\left(\frac{W_{u_1}+W_{y_1}}{(2m)^{1/2}\mu_{Y_1}}+\frac{W_{u_2}+W_{y_2}}{(2m)^{1/2}\mu_{Y_2}}\right)H_1+\frac{W_{v_1}+W_{x_1}}{(2m)^{1/2}\mu_1}H_2$$

$$+\frac{W_{v_2}+W_{x_2}}{(2m)^{1/2}\mu_2}H_3-\frac{1}{2}\sum_{i=1}^{2}\Bigg\{\left(\frac{W_{u_1}^{(i)}+W_{y_1}^{(i)}}{m^{1/2}\mu_{Y_1}}+\frac{W_{u_2}^{(i)}+W_{y_2}^{(i)}}{m^{1/2}\mu_{Y_2}}\right)$$

$$+\frac{W_{v_1}^{(i)}+W_{x_1}^{(i)}}{m^{1/2}\mu_1}H_2+\frac{W_{v_2}^{(i)}+W_{x_2}^{(i)}}{m^{1/2}\mu_2}H_3\Bigg\}\Bigg]^2$$

$$=\frac{P^2}{2m}\Bigg[\left(\frac{1}{\mu_{Y_2}^2}\sigma_{Y_1}^2+\frac{1}{\mu_{Y_2}^2}\sigma_{Y_2}^2+\frac{1}{\mu_{Y_1}\mu_{Y_2}}\sigma_{Y_1Y_2}\right)H_1^2$$

$$+\left(\frac{1}{P^2\mu_1^2}\sigma_{X_1}^2H_2^2+\frac{1}{P^2\mu_2^2}\sigma_{X_2}^2H_3^2+\frac{2}{P^2\mu_1\mu_2}\sigma_{X_1X_2}H_2H_3\right)$$

$$+\frac{2}{\mu_{Y_1}}\left(\frac{1}{\mu_1}\sigma_{X_1Y_1}H_1H_2+\frac{1}{\mu_2}\sigma_{X_2Y_1}H_1H_3\right)$$

$$+\frac{2}{\mu_{Y_2}}\left(\frac{1}{\mu_1}\sigma_{X_1Y_2}H_1H_2+\frac{1}{\mu_2}\sigma_{X_2Y_2}H_1H_3\right)\Bigg]$$

$$+\frac{P}{2m}\Bigg[\left(\frac{1}{\mu_{Y_1}^2}\sigma_u^2+\frac{1}{\mu_{Y_2}^2}\sigma_u^2\right)H_1^2$$

$$+\left(\frac{1}{P^2\mu_1^2}\sigma_{X_1}^2H_2^2+\frac{1}{P^2\mu_2^2}\sigma_{X_2}^2H_3^2\right)\Bigg] \quad (4.17)$$

5 CONCLUDING REMARK

For the proposed generalized estimators $\hat{P}_h$ and $\hat{P}_{Hj}$, it can be easily seen from (3.2), (3.3), (4.16) and (4.17) that bias and the MSE are given by

$$\text{Bias}\left(\hat{P}_h\right)=\frac{P}{n}\left(\frac{1}{2\mu_1^2}\sigma_{x_1}^2h_{030}+\frac{1}{2\mu_2^2}\sigma_{x_2}^2h_{003}+\frac{1}{\mu_P\mu_1\mu_2}\sigma_{(x_1,x_2)}h_{011}\right.$$

$$+\frac{1}{\mu_{Y_1}}\left(\frac{1}{\mu_1}\sigma_{(x_1,y_1)}h_{110}+\frac{1}{\mu_2}\sigma_{(x_2,y_1)}h_{101}\right)$$
$$+\frac{1}{\mu_{Y_2}}\left(\frac{1}{\mu_1}\sigma_{(x_1,y_2)}h_{110}+\frac{1}{\mu_2}\sigma_{(x_2,y_2)}h_{101}\right)\Bigg]$$
$$+\frac{P}{n}\left[\frac{1}{2\mu_1^2}\sigma^2_{v_1}h_{030}+\frac{1}{2\mu_2^2}\sigma^2_{v_2}h_{003}\right] \qquad (5.1)$$

$$MSE\left(\hat{P}_h\right)=\frac{P^2}{n}\Bigg[\left(\frac{1}{\mu^2_{Y_2}}\sigma^2_{y_1}+\frac{1}{\mu^2_{Y_2}}\sigma^2_{y_2}+\frac{1}{\mu_{Y_1}\mu_{Y_2}}\sigma_{(y_1,y_2)}\right)h_1^2$$
$$+\left(\frac{1}{P^2\mu_1^2}\sigma^2_{x_1}h_2^2+\frac{1}{P^2\mu_2^2}\sigma^2_{x_2}h_3^2+\frac{2}{P\mu_1\mu_2}\sigma_{(x_1,x_2)}h_2h_3\right)$$
$$+\frac{2}{\mu_{Y_1}}\left(\frac{1}{\mu_1}\sigma_{(x_1,y_1)}h_1h_2+\frac{1}{\mu_2}\sigma_{(x_2,y_1)}h_1h_3\right)$$
$$+\frac{2}{\mu_{Y_2}}\left(\frac{1}{\mu_1}\sigma_{(x_1,y_2)}h_1h_2+\frac{1}{\mu_2}\sigma_{(x_2,y_2)}h_1h_3\right)\Bigg]$$
$$+\frac{P}{n}\Bigg[\left(\frac{1}{\mu^2_{Y_2}}\sigma^2_{u_1}+\frac{1}{\mu^2_{Y_2}}\sigma^2_{u_2}\right)h_1^2$$
$$+\left(\frac{1}{P^2\mu_1^2}\sigma^2_{x_1}h_2^2+\frac{1}{P^2\mu_2^2}\sigma^2_{x_2}h_3^2\right)\Bigg] \qquad (5.2)$$

$$Bias\left(\hat{P}_{Hj}\right)=0 \qquad (5.3)$$

$$MSE\left(\hat{P}_{Hj}\right)=\frac{P^2}{2m}\Bigg[\left(\frac{1}{\mu^2_{Y_2}}\sigma^2_{y_1}+\frac{1}{\mu^2_{Y_2}}\sigma^2_{y_2}+\frac{1}{\mu_{Y_1}\mu_{Y_2}}\sigma_{(y_1,y_2)}\right)H_1^2$$
$$+\left(\frac{1}{P^2\mu_1^2}\sigma^2_{x_1}H_2^2+\frac{1}{P^2\mu_2^2}\sigma^2_{x_2}H_3^2+\frac{2}{P\mu_1\mu_2}\sigma_{(x_1,x_2)}H_2H_3\right)$$
$$+\frac{2}{\mu_{Y_1}}\left(\frac{1}{\mu_1}\sigma_{(x_1,y_1)}H_1H_2+\frac{1}{\mu_2}\sigma_{(x_2,y_1)}H_1H_3\right)$$
$$+\frac{2}{\mu_{Y_2}}\left(\frac{1}{\mu_1}\sigma_{(x_1,y_2)}H_1H_2+\frac{1}{\mu_2}\sigma_{(x_2,y_2)}H_1H_3\right)\Bigg]$$
$$+\frac{P}{2m}\Bigg[\left(\frac{1}{\mu^2_{Y_2}}\sigma^2_{u_1}+\frac{1}{\mu^2_{Y_2}}\sigma^2_{u_2}\right)H_1^2$$
$$+\left(\frac{1}{P^2\mu_1^2}\sigma^2_{v_1}H_2^2+\frac{1}{P^2\mu_2^2}\sigma^2_{v_2}H_3^2\right)\Bigg] \qquad (5.4)$$

From (5.2) and (5.4), we see that both estimators $\hat{P}_h$ and $\hat{P}_{Hj}$ have the same mean square error but from (5.1) bias of $\hat{P}_h$ is not zero whereas from (5.3) bias of the jack-knifed estimator $\hat{P}_{Hj}$ is zero. Hence, both the estimators $\hat{P}_h$ and $\hat{P}_{Hj}$ having the same mean square error, but the jack-knifed estimator $\hat{P}_{Hj}$ may be preferred to the estimator $\hat{P}_h$ in the sense of unbiasedness under the presence of measurement errors.

Therefore, the proposed unbiased jack-knifed estimator should be preferred than the conventional estimators under measurement error as they help in removing the bias while still preserving the efficiency. Many important estimators like adapted estimators of Srivastava (1967) type for estimation of ratio of population means given by

$$\hat{P}_\alpha=\hat{P}\left(\frac{\bar{x}_1}{\mu_1}\right)^{\alpha_1}\left(\frac{\bar{x}_2}{\mu_2}\right)^{\alpha_2}$$

are special cases of the present study when we assume the measurement errors to be absent. Such conventional results can be obtained as special cases of this study by setting the measurement error variances to be zero or by choosing the derivatives judiciously under measurement errors.

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