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Computational Approach in Testing of Hypothesis for Generalized Exponential Distribution .

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Abstract

In this paper we provide a step by step computational approach to handle statistical inferences based on a parametric model for a given data set. Computational approach may come handy in those cases where the sampling distributions are not easy to derive or extremely complicated. This approach provides an algorithmic framework based on the Monte Carlo simulation and numerical computations which can be implemented mechanically by applied researchers to draw statistical inferences when a suitable parametric model is assumed for a given data set. We applied to two real life data sets to show how easily it can be implemented, and in terms of power it can be as good as (if not better than) the other reported method(s).

Key words:

1 Introduction

Recent advances in mathematical and statistical packages have made a tremendous positive impact on fundamental science's research. This is more so true in statistical research where conventional wisdom now demands that theoretical results be supported by numerical results to justify their usefulness. The numerical results are easy to obtain with the help of superior computational resources. The motivation of this paper comes from the need for making statistical tools more readily available to the applied researchers, especially in the areas of life testing and reliability. Here we concentrate only on testing of hypothesis. In hypothesis testing, one attempts to answer the following question: If the null hypothesis is assumed to be true, what is the probability of obtaining the observed result, or any more extreme result that is favourable to the alternative hypothesis?

Suppose we have independent and identically distributed (i.i.d.) observations $X_1, X_2, ..., X_n$. In parametric statistics one assumes a family $F = f(x, \theta), \theta \in \Theta$, of probability distributions for the data. The parameter space Θ is usually taken as the natural one (i.e., collection of all possible values of θ for which $f(x_j, \theta)$ is a pdf or pmf), and the structure of f is assumed to be known except the unknown parameter θ (which can be vector valued). The statistical inference about the population from which the observations $X_1, X_2, ..., X_n$ have been drawn revolves around the unknown parameter θ.

Any inference on $\hat{\theta}$ starts with a point estimator $\hat{\theta} = \hat{\theta}(X_1, X_2, ..., X_n)$, a rationally guessed value of θ based on the data $X_1, X_2, ..., X_n$. Since the data itself is random, so is the estimator $\hat{\theta}$, which brings another level of uncertainty in our decision making process. To evaluate the performance of $\hat{\theta}$ one needs to look at the probability distribution of $\hat{\theta}$, commonly known as the sampling distribution, which may not be easy to derive. In such a situation one looks at the asymptotic distribution (where the probability distribution of a standardized version of $\hat{\theta}$ is derived as n, the sample size, approaches to ∞) , or studies the distribution of $\hat{\theta}$ through simulation where repeated set of observations are generated from $f(x_j, \theta)$ for different values of $\theta \in \Theta$. We follow the latter approach in this paper in a more detailed manner.

2 Generalized Exponential Distribution

Generalized Exponential (GE) Distribution has been introduced by the authors (Gupta and Kundu, 1999a). The GE distribution has the distribution function;

$$F_0(t) = \begin{cases} (1 - e^{-\lambda t})^{\theta} & ; \ t > 0, \theta > 0, \lambda > 0 \\ 0 & ; \ otherwise. \end{cases} \tag{2.1}$$

and

$$f_0(t) = \begin{cases} \theta \lambda e^{-\lambda t}(1 - e^{-\lambda t})^{\theta - 1} & ; \ t > 0, \theta > 0, \lambda > 0 \\ 0 & ; \ otherwise. \end{cases} \tag{2.2}$$

where λ and θ are respectively scale and shape parameters of the distribution. Estimation of θ and λ when both are unknown. If $x_1, x_2, ..., x_n$ is a random sample from $GE(\lambda, \theta)$, then the log-likelihood function, $L(\lambda, \theta)$, is

$$L(\lambda, \theta) = nln(\theta) + nln(\lambda) + (\theta - 1)\sum_{i=1}^{n} ln(1 - e^{-\lambda x_i}) - \lambda \sum_{i=1}^{n} x_i \tag{2.3}$$

The normal equations become:

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} ln(1 - e^{-\lambda x_i}) = 0 \tag{2.4}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + (\theta - 1)\sum_{i=1}^{n} \frac{x_i e^{-x_i}}{(1 - e^{-\lambda x_i})} + \sum_{i=1}^{n} x_i = 0 \tag{2.5}$$

From (2.4), we obtain the MLE ofθ as a function of λ, say $\hat{\theta}(\lambda)$, where

$$\hat{\theta}(\lambda) = -\frac{n}{\sum_{i=1}^{n} ln(1 - e^{-\lambda x_i})} \tag{2.6}$$

Putting $\hat{\theta}(\lambda)$ in (2.3), we obtain

$$g(\lambda) = L(\hat{\theta}(\lambda), \lambda) = C - nln\sum_{i=1}^{n}(-ln(1 - e^{-\lambda x_i})) \tag{2.7}$$
$$+nln(\lambda) - \sum_{i=1}^{n} ln(1 - e^{-\lambda x_i}) - \lambda \sum_{i=1}^{n} x_i$$

Therefore, MLE of λ , say $\hat{\lambda}_{MLE}$, can be obtained by maximizing (2.7) with respect to λ. It is observed in Gupta and Kundu (1999b) that $g(\lambda)$ is a unimodal function and the$\hat{\lambda}_{MLE}$ which maximizes (2.7) can be obtained from the fixed point solution of

$$h(\lambda) = \lambda \tag{2.8}$$

where

$$h(\lambda) = \left[\frac{\sum_{i=1}^{n}((x_i e^{-x_i})/(1 - e^{-\lambda x_i}))}{\sum_{i=1}^{n} ln(1 - e^{-\lambda x_i})} + \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{(1 - e^{-\lambda x_i})} \right]^{-1} \tag{2.9}$$

Very simple iterative procedure can be used to find a solution of (2.9) and it works very well. Once we obtain $\hat{\lambda}_{MLE}$, the MLE of θ say $\hat{\theta}_{MLE}$ can be obtained from (2.6) as $\hat{\theta}_{MLE} = \hat{\theta}(\hat{\lambda}_{MLE})$.

To obtain the asymptotic normality results, obtain the asymptotic variances of the different parameters. It can be stated as follows:

$$\left[\sqrt{n}(\hat{\theta}_{MLE} - \theta), \sqrt{n}(\hat{\lambda}_{MLE} - \lambda)\right] \to N_2\left(0, I^{-1}(\theta, \lambda)\right) \tag{2.10}$$

where $I(\theta, \lambda)$ is the Fisher Information matrix, i.e.,

$$I(\theta, \lambda) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 L}{\partial \theta \partial \lambda}\right) \\ E\left(\frac{\partial^2 L}{\partial \lambda \partial \theta}\right) & E\left(\frac{\partial^2 L}{\partial \lambda^2}\right) \end{bmatrix} \tag{2.11}$$

So we can obtain the asymptotic normality results. The exact joint distribution of $(\hat{\theta}, \hat{\lambda})$ is not possible to derive analytically, nor can it be visualized easily through simulation. So in the next section we provide the general computational framework to handle a scalar valued parameter based on the maximum likelihood estimator. The justification is given by showing how the method works for the well known normal distribution.

3 Hypothesis Testing Based on a Computational Approach

The proposed hypothesis testing based on computational approach (CA) test based on simulation and numerical computations uses the ML estimate(s), but doesn't require any asymptotic distribution. The CA is given through the following steps.

3.1 Case I

Let θ is scalar valued and there is no nuisance parameter. Our goal is to test $H_0 : \theta = \theta_0$ against a suitable $H_1(\theta < \theta_0$ or $\theta > \theta_0$ or $\theta \neq \theta_0)$ at level α.

Method

Step I- Obtain $\hat{\theta}_{MLE}$, the MLE of θ.

Step II- Set $\theta = \theta_0$ (specified by H_0). Generate artificial samples $X = (X_1, X_2, ..., X_n)$

of size n iid from $f(x_j, \theta_0)$ a large number of times (say, M times). For each of these replicated samples recalculate the MLE of θ (assuming that it is unknown). For the k^{th} replicated sample $X(k) = (X(k)_1, X(k)_2, ..., X(k)_n)$ (say) iid $f(x_j, \theta_0)$, the recalculated MLE of θ is $\hat{\theta}_{0k}$ (i.e., $\hat{\theta}_{0k}$ is the value of $\hat{\theta}_{MLE}$ based on $X(k)$, $k = 1, 2, ..., M$. Then order these simulated MLEs as $\hat{\theta}_{0(1)}, \hat{\theta}_{0(2)}, ..., \hat{\theta}_{0(M)}$.

Step III- (i) For testing H_0 against $H_1 : \theta < \theta_0$; define $\hat{\theta}_L = \hat{\theta}_{0(\alpha M)}$. Reject H_0 if $\hat{\theta}_{MLE} < \hat{\theta}_L$; and accept H_0 if H_0 $\hat{\theta}_{MLE} \geq \hat{\theta}_L$. Alternatively, calculate the $p-$value as : $p-$value $=$ (number of $\hat{\theta} < \hat{\theta}_{MLE})/M$. (ii) For testing H_0 against $H_1 : \theta > \theta_0$, define $\hat{\theta}_U = \hat{\theta}_{0((1-\alpha)M)}$. Reject H_0 if H_0 $\hat{\theta}_{MLE} > \hat{\theta}_U$; and accept H_0 if H_0 $\hat{\theta}_{MLE} \leq \hat{\theta}_U$. Alternatively, calculate the $p-$value as : $p-$value $=$ (number of $\hat{\theta} > \hat{\theta}_{MLE})/M$. (iii) For testing H_0 against $H_1 : \theta \neq \theta_0$, define $\hat{\theta}_U = \hat{\theta}_{0((1-\alpha/2)M)}$ and $\hat{\theta}_L = \hat{\theta}_{0(\alpha/2M)}$. Reject H_0 if $\hat{\theta}_{MLE}$ is either greater than $\hat{\theta}_U$ or less than $\hat{\theta}_L$; accept H_0 if otherwise. Alternatively the $p-$value is computed as: $p-$value $= 2min(p1; p2)$, where $p1 =$ (number of $\hat{\theta} < \hat{\theta}_{MLE})/M$ and $p2 =$ (number of $\hat{\theta} > \hat{\theta}_{MLE})/M$.

3.2 Application to Normal Distribution

As an application of the above Case-1, consider the problem of testing $H_0 : \mu = \mu_0$ against$H_1 : \mu \neq \mu_0$, where $X_1, X_2, ..., X_n$ are from $N(\mu; 1), \mu \in R$.

Step-1, we have $\hat{\mu} = \bar{X}$ (which has $N(\mu, 1/n)$ distribution).

Step-2, we fix $\mu = \mu_0$ (H_0 value), and generate

(1st replication): $(X(1)_1, X(1)_2, ..., X(1)_n)$ iid $N(\mu_0; 1)$; get $\hat{\mu}_{01} = \bar{X}(1)$

(2nd replication): $(X(2)_1, X(2)_2, ..., X(2)_n)$ iid $N(\mu_0; 1)$; get $\hat{\mu}_{02} = \bar{X}(2)$

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(Mth replication): $(X(M)_1, X(M)_2, ..., X(M)_n)$ iid $N(\mu_0; 1)$; get $\hat{\mu}_{0M} = \bar{X}(M)$

[Note that the values $\hat{\mu}_{01}, \hat{\mu}_{02}, ..., \hat{\mu}_{0M}$ are representing the $N(\hat{\mu}_0, 1/n)$ distribution; i.e., a relative frequency histogram of these values is closely matched by the $N(\mu_0, 1/n)$ pdf for sufficiently large M.] The values $\hat{\mu}_{01}, \hat{\mu}_{02}, ..., \hat{\mu}_{0M}$ are ordered as $\hat{\mu}_{0(1)}, \hat{\mu}_{0(2)}, ..., \hat{\mu}_{0(M)}$.

Step-3, obtain $\hat{\mu}_L = \hat{\mu}_{(0(\alpha/2)M)}$ and $\hat{\mu}_U = \hat{\mu}_{(0(1-\alpha/2)M)}$. [Since M is sufficiently large, $\hat{\mu}_L \approx (\mu_0 - z_{\alpha/2}/\sqrt{n})$ and $\hat{\mu}_U \approx (\mu_0 + z_{\alpha/2}/\sqrt{n})$ where $z_{\alpha/2}$ is the right tail $\alpha/2$-probability cut-off point of the standard normal distribution. Therefore, CA is identical to the classical test procedure for the above normal distribution.]

3.3 Case 2

If nuisance parameter is present. Assume that $\theta = (\theta^{(1)}, \theta^{(2)}) \in \Theta$, where $\theta^{(1)}$ is the scalar valued parameter of interest, and $\theta^{(2)}$ is the nuisance parameter which can be vector valued. Our goal is to test $H_0 : \theta^{(1)} = \theta_0^{(1)}$ against a suitable $H_1(\theta^{(1)} < \theta_0^{(1)}$ or $\theta^{(1)} > \theta_0^{(1)}$ or $\theta^{(1)} \neq \theta_0^{(1)})$ at level α. The following steps are a slight generalization of those discussed under the previous case.

Method:

Step 1- Obtain $\hat{\theta}_{MLE} = (\hat{\theta}^{(1)}_{MLE}, \hat{\theta}^{(2)}_{MLE})$, the MLE of θ

Step 2- (i) Set $\theta^{(1)} = \theta_0^{(1)}$, then find the MLE of $\theta^{(2)}$ from the original data, and call this as the restricted MLE of $\theta^{(2)}$, denoted by $\hat{\theta}^{(2)}_{RMLE}$.

(ii) Generate artificial sample $X = (X_1, X_2, ..., X_n)$ iid from $f(x_j|\theta_0^{(1)}, \hat{\theta}^{(2)}_{RMLE})$ a large number of times (say, M times). For each of these replicated samples, recalculate the MLE of $\theta = (\theta^{(1)}, \theta^{(2)})$ (pretending that it were unknown), and retain only the first component that is relevant for $\theta^{(1)}$. Let these recalculated MLE values of $\theta^{(1)}$ be $\theta^{(1)}_{01}, \theta^{(1)}_{02}, ..., \theta^{(1)}_{0M}$.

(iii) Let $\theta^{(1)}_{0(1)}, \theta^{(1)}_{0(2)}, ..., \theta^{(1)}_{0(M)}$ be the ordered values of $\theta^{(1)}_{0(k)}, 1 \leq k \leq M$.

Step 3- Almost similar to that of Case-1, with the exception that $\theta^{(1)}_{0(k)}$ ' s are used instead of $\theta_{0(k)}$ and $\hat{\theta}^{(1)}_{MLE}$ is used in place of $\hat{\theta}_{MLE}$.

3.4 Application to Normal Distribution

Again, as an application of the above Case-2, consider the problem of testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$when we have iid observations $X_1, X_2, ..., X_n$ from $N(\mu, \sigma^2)$ where both the parameters are unknown.

Step 1- We have $\hat{\mu}_{MLE} = \bar{X}$ and $\hat{\sigma^2}_{MLE} = S/n$ (where $S = \sum_{i=1}^{n}(X_i - \bar{X})).\hat{\mu}_{MLE}$is distributed as$N(\mu, \sigma^2/n)$, which is independent of $\hat{\sigma^2}_{MLE}$. Note that $\hat{\sigma^2}_{MLE}$ is distributed as $(\sigma^2/n)\chi^2_{(n-1)}$.

Step 2-

(i) Set $\mu = \mu_0$; i.e.,$X_1, X_2, ..., X_n$ are iid $N(\mu_0, \sigma^2)$ where σ^2 is the only unknown parameter. The restricted MLE of σ^2 is $\sigma^2_{RMLE} = \sum_{i=1}^{n}(X_i - \mu_0) = (S/n) + (\bar{X} - \mu_0)^2$

(ii) Generate samples from $N(\mu_0, \hat{\sigma}^2_{RMLE})$as

(1st replication): $(X(1)_1, X(1)_2, ..., X(1)_n)$ iid $N(\mu_0; \hat{\sigma}^2_{RMLE})$; get $\hat{\mu}_{01} = \bar{X}(1)$

(2nd replication): $(X(2)_1, X(2)_2, ..., X(2)_n)$ iid $N(\mu_0; \hat{\sigma}^2_{RMLE})$; get $\hat{\mu}_{02} = \bar{X}(2)$

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(Mth replication): $(X(M)_1, X(M)_2, ..., X(M)_n)$ iid $N(\mu_0; \hat{\sigma}^2_{RMLE})$; get $\hat{\mu}_{0M} = \bar{X}(M)$

(Note that the values $\hat{\mu}_{ok}, 1 \leq k \leq M$ follow $N(\mu_0; \hat{\sigma}^2_{RMLE}/n)$ distribution, i.e., a relative frequency histogram of these $\hat{\mu}_{01}, \hat{\mu}_{02}, ..., \hat{\mu}_{0M}$ should resemble the pdf closely.)

(iii) Obtain $\hat{\mu}_{0(1)}, \hat{\mu}_{0(2)}, ..., \hat{\mu}_{0(M)}$ from $\hat{\mu}_{01}, \hat{\mu}_{02}, ..., \hat{\mu}_{0M}$. Step 3- The lower and upper $(\alpha/2)$ probability cut-off points are obtained as $\hat{\mu}_L = \hat{\mu}_{(0(\alpha/2)M)}$ and $\hat{\mu}_U = \hat{\mu}_{(0(1-\alpha/2)M)}$ which are essentially $(\mu_0 - \hat{\sigma}_{RMLE} z_{(\alpha/2)}/\sqrt{n})$ and $(\mu_0 + \hat{\sigma}_{RMLE} z_{(\alpha/2)}/\sqrt{n})$ respectively (since M is sufficiently large).

Let us now see the implications of the CA for the normal distribution with unknown variance. The rejection region for H_0 is $\hat{\mu}_{MLE} > \hat{\mu}_U or \hat{\mu}_{MLE} < \hat{\mu}_L$, i.e, reject H_0 if (approximately) $|\bar{X} > \mu_0 + \hat{\sigma}_{RMLE} z_{(\alpha/2)}/\sqrt{n})$ or $|\bar{X} < \mu_0 - \hat{\sigma}_{RMLE} z_{(\alpha/2)}/\sqrt{n})$ i.e. $|\sqrt{n}(\bar{X} - \mu_0)| > \hat{\sigma}_{RMLE} z_{(\alpha/2)}$

Now we show that how new critical region (in the last paragraph) compare with the traditional critical region $|\sqrt{n}(\bar{X} - \mu_0)| > \sqrt{S/(n-1)}t_{(n-1),(\alpha/2)}$ of the usual t-test?

So we have now two competing critical regions, one based on the CA, i.e.,

$$R_{CA} : |\sqrt{n}(\bar{X} - \mu_0)| > \hat{\sigma}_{RMLE} z_{(\alpha/2)}$$

i.e.

$$R_{CA} : |\sqrt{n}(\bar{X} - \mu_0)| > \sqrt{S/(n-1)}| > \sqrt{((n-1)/n) + (\bar{X} - \mu_0)^2/(S/(n-1))} z_{(\alpha/2)} \quad (3.1)$$

and the other that is based on the usual t-test, i.e.,

$$R_T : |\sqrt{n}(\bar{X} - \mu_0)/\sqrt{S/(n-1)}| > t_{(n-1),(\alpha/2)} \quad (3.2)$$

where $t_{(n-1),(\alpha/2)}$ is the right tail $(\alpha/2)$ probability cut-off point of the $t_{(n-1)}$ -distribution. In the following we compare the critical regions (3.1) and (3.2).

The test statistic $|\Delta| = |\sqrt{n}(\bar{X} - \mu_0)/\sqrt{S/(n-1)}|$ of our usual t-test has a fixed cut-off point $t_{(n-1),(\alpha/2)}$ whereas in our suggested CA, $|\Delta|$ has a stochastic cut-off point. Taking $(n-1)/n \approx 1$, the right hand side (RHS) of (2.1) is slightly larger than $z_{(\alpha/2)}$, but not necessarily larger than $t_{(n-1),(\alpha/2)}$, the RHS of (2.2).

The inequality in (3.1) can be simplified further as

$$|\Delta| > \sqrt{((n-1)/n) + \Delta^2/n} z_{(\alpha/2)}$$

$$i.e. \Delta^2(1 - z^2_{(\alpha/2)/n}) > ((n-1)/n) z^2_{(\alpha/2)} \quad (3.3)$$

Assume that

$$(1 - z^2_{(\alpha/2)/n}) > 0 \quad (3.4)$$

then (2.3) is equivalent to

$$\Delta^2 > ((n-1)/n) z^2_{(\alpha/2)}/(1 - z^2_{(\alpha/2)/n}) \quad (3.5)$$

Thus, the size of our test procedure (3.1) is P(eq(3:5) holds$|\mu = \mu_0$) and this is calculated for some selected values of α and n in the following table (Table 2.1). Note that under H_0, $\Delta^2 \sim F_{1,(n-1)}$.

Interestingly, from Table 1, for $\alpha = 0.10$ our test based on CA has size almost 0.10; but for $\alpha = 0.05$, our test is slightly conservative. This is a small price to pay for not

Table 1: Exact Size of the CA When the Usual Test (t-Test) has Size α

n		5	10	15	20	25	35	50	75	100
$\alpha = 0.10$	$z_{(\alpha/2)} = 1:6448$	0.0956	0.1010	0.1011	0.1009	0.1008	0.1006	0.1005	0.1003	0.1002
$\alpha = 0.05$	$z_{(\alpha/2)} = 1:9600$	0.0220	0.0420	0.0455	0.0470	0.0476	0.0484	0.0489	0.0493	0.0495

using the exact sampling distribution. As it seems, the square root of the right hand side of (2.5) is not very different from $t_{(n-1),(\alpha/2)}$. Thus, we have an approximation of the t-cut off points

$$t_{(n-1),(\alpha/2)} \approx ((n-1)/n)z^2_{(\alpha/2)}/(1 - z^2_{(\alpha/2)/n}) \tag{3.6}$$

provided the condition (2.4) holds, i.e.,$z_{(\alpha/2)} < \sqrt{n}$. When z(=2) $z_{(\alpha/2)} \geq \sqrt{n}$, that means when the sample size is very small (less than 4 or 5), then the size of the CA is 0. But for such small sample sizes, any simulation would be unreliable. In (3.6), the = holds as $n \to \infty$. Also, the power of our CA is very close to the one based on the usual t-test.

Remark The applications of our suggested CA to the above normal distributions are quite interesting. From the above two remarks what we have seen is that though the normal theory provides us a nice simple (and optimal in terms of power and unbiasedness) test, one can do almost as well just by following our computational steps mechanically. There is no need to know the sampling distributions of $\bar{X}$ and/or S, and their mutual independence. It is this usefulness which motivates us to look for applications of this computational approach for model beyond the normal one.

4 Data Analysis

In this section we use two uncensored data sets and it fits well for generalized exponential distribution.

Data Set 1: The first data set is as follows; (Lawless, 1986 page 228). The data given here arose in tests on endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.40.

We have fitted generalized exponential distribution to this data set. We present the estimates, the Log-likelihood (LL), the observed and the expected values and the χ^2 statistics. The results are as follows. $\hat{\lambda} = 0.0314; \hat{\alpha} = 5.2589; LL = -112.9763; \chi^2 = 0.783$.

We now apply the CA discussed in Section 3 to test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$. The same can be followed for testing $\theta = \theta_0$, for any $\theta_0 (> 0)$ other than 1. $\theta = 1$ is of particular interest since it reduces the model to a much convenient exponential distribution.

4.1 Test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ at $\alpha = 0.05$

Step 1- Obtain $\hat{\lambda} = 0.03229609$ and $\hat{\alpha} = 5.283211$ by using (2.4) and (2.9).

Step 2- (i) Set $H_0 : \theta = 1$ (H_0 value). Then find the restricted MLE of λ (with H_0 restriction on θ) as $\hat{\lambda} = 0.01384308$.

(ii) Now Generate artificial sample $X = (X_1, X_2, ..., X_n)$ iid from $GE(0.01384308, 1)$ a large number of times (say, M times). For each of these replicated samples, recalculate the MLE of $\hat{\lambda}, \hat{\theta}$ (pretending that it were unknown), and retain only the second component that is relevant for $\hat{\theta}$. Let these recalculated MLE values of $\hat{\theta}_{MLE}$ be values denoted as $\hat{\theta}_{01}, \hat{\theta}_{02}, ..., \hat{\theta}_{0M}$ ordered .

(iii) Order the above $\hat{\theta}_{MLE}$ values from replications and get $\hat{\theta}_{0(1)}, \hat{\theta}_{0(2)}, ..., \hat{\theta}_{0(M)}$. Using the level of significance $\alpha = 0.05$, the lower and upper cut off points are obtained as $\hat{\theta}_L = \hat{\theta}_{0(2500)} = 0.6894382$ and $\hat{\theta}_U = \hat{\theta}_{0(97500)} = 1.579667$.

Step 3- Since $\hat{\theta}_{MLE} = 5.283211$ falls outside the two cut-off points given above, we reject the $H_0 : \theta = 1$ at 5% level.

4.2 Power for testing $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ at $\alpha = 0.05$

In the following the power of the above CA is computed through simulation, and this has been done for n = 23 (to comply with the dataset 1). The power computation is done through the following steps.

Step 1 For fixed $n(= 23)$, $\lambda = 1$ and $\theta = (= 0.1, ..., 5.0)$, generate i.i.d. observations of size from $GE(\lambda, \theta)$.

Step 2- Get $\hat{\lambda}$ and $\hat{\theta}$.

Step 3- Set $\theta = 1$ (H_0 value), and get the restricted MLE of λ as $\hat{\lambda}_{RMLE}$.
Step 4- Now generate $X^{(l)} = (X_1^{(l)}, X_2^{(l)}, ..., X_n^{(l)})$ i.i.d. from $GE(\lambda = \hat{\lambda}_{RMLE}, \theta = 1)$; $l = 1, 2, ..., M(= 5,000)$: Retain only the MLE values of θ as $\hat{\theta}_{01}, \hat{\theta}_{02}, ..., \hat{\theta}_{0M}$. Order these MLE values of θ as $\hat{\theta}_{0(1)}, \hat{\theta}_{0(2)}, ..., \hat{\theta}_{0(M)}$ Get $\hat{\theta}_U = \hat{\theta}_{0(4875)}$ and $\hat{\theta}_L = \hat{\theta}_{0(125)}$ (these are the lower and upper 2.5% cut off points).
Step 5- Now bring the $\hat{\theta}_{MLE}$ from the above Step 2, and get $I = 1I(\theta_L \leq \hat{\theta} \leq \theta_U)$.
Step 6- Repeat the above Step 1 through Step 5 a large number of times (say, N times), and get the I values as $I_1, I_2, ..., I_N$. Finally, the power is approximated as

$$\beta_{CA}(\theta) = \sum_{i=1}^{N} I_i, (N = 5000) \tag{4.1}$$

Table 2: Power of the CA for two sided alternative $n = 23, \alpha = 0.05$

θ	0.25	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Power	1.000	0.996	0.964	0.855	0.615	0.406	0.251	0.132
θ	1.1	1.3	1.5	1.7	2	2.5	3.0	3.5
Power	0.180	0.347	0.518	0.670	0.833	0.972	0.995	0.997

Data set 2: (Linhart and Zucchini (1986, page 69). The following data are failure times of the air conditioning system of an airplane: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95.
We have fitted generalized exponential distribution to this data set. We present the estimates, the Log-likelihood (LL), the observed and the expected values and the χ^2 statistics. The results are as follows. $\hat{\lambda} = 0.01454297; \hat{\alpha} = 0.8092874; LL = -152.264; \chi^2 = 3.383$
We now apply the CA discussed in Section 3 to test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$. The same can be followed for testing $\theta = \theta_0$, for any $\theta_0 (> 0)$ other than 1. $\theta = 1$ is of particular interest since it reduces the model to a much convenient exponential distribution.

4.3 Test $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ at $\alpha = 0.05$

Step 1- Obtain $\hat{\lambda} = 0.01454297$ and $\hat{\alpha} = 0.8092874$ by using (2.4) and (2.9).
Step 2- (i) Set $H_0 : \theta = 1$ (H_0 value). Then find the restricted MLE of λ (with H_0 restriction on θ) as $\hat{\lambda} = 0.01677852$.

(ii) Now Generate artificial sample $X = (X_1, X_2, ..., X_n)$ iid from $GE(0.01677852, 1)$ a large number of times (say, M times). For each of these replicated samples, recalculate the MLE of $\hat{\lambda}, \hat{\theta}$ (pretending that it were unknown), and retain only the second component that is relevant for $\hat{\theta}$. Let these recalculated MLE values of $\hat{\theta}_{MLE}$ be values denoted as $\hat{\theta}_{01}, \hat{\theta}_{02}, ..., \hat{\theta}_{0M}$ ordered .

(iii) Order the above $\hat{\theta}_{MLE}$ values from replications and get $\hat{\theta}_{0(1)}, \hat{\theta}_{0(2)}, ..., \hat{\theta}_{0(M)}$. Using the level of significance $\alpha = 0.05$, the lower and upper cut off points are obtained as $\hat{\theta}_L = \hat{\theta}_{0(2500)} = 0.7187328$ and $\hat{\theta}_U = \hat{\theta}_{0(97500)} = 1.579667$.

Step 3- Since $\hat{\theta}_{MLE} = 0.01677852$ falls within the two cut-off points given above, we accept the $H_0 : \theta = 1$ at 5% level.

Table 3: Power of the CA for two sided alternative $n = 23, \alpha = 0.05$

θ	0.25	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Power	1.000	0.998	0.985	0.91	0.734	0.469	0.297	0.169
θ	1.1	1.3	1.5	1.7	2	2.5	3.0	3.5
Power	0.187	0.381	0.619	0.762	0.918	0.989	1	1

5 Conclusion

The whole idea behind this work has been to show that with the availability of cheap and fast computing facilities, statistical inference can be done solely based on computations and simulation without going through the complicated analytic derivations of the sampling distributions. Some / Many times it not possible to derive the exact distribution of test statistics. In those cases this approach is very handy. We give the details of used method and also apply the method to two real life data. One data reject the H_0 and in second data accept the H_0. We also found that as sample size increases the power of test also increases. The computational approach (CA) adopted in this work clearly shows that, at least for the examples used here, no significant compromise has been made in terms of level or size as far as testing is concerned. As a result, applied researchers can adopt the CA without getting involved into analytical complexities. This work can be extended in Bayesian set-up.

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Appendix : Figures

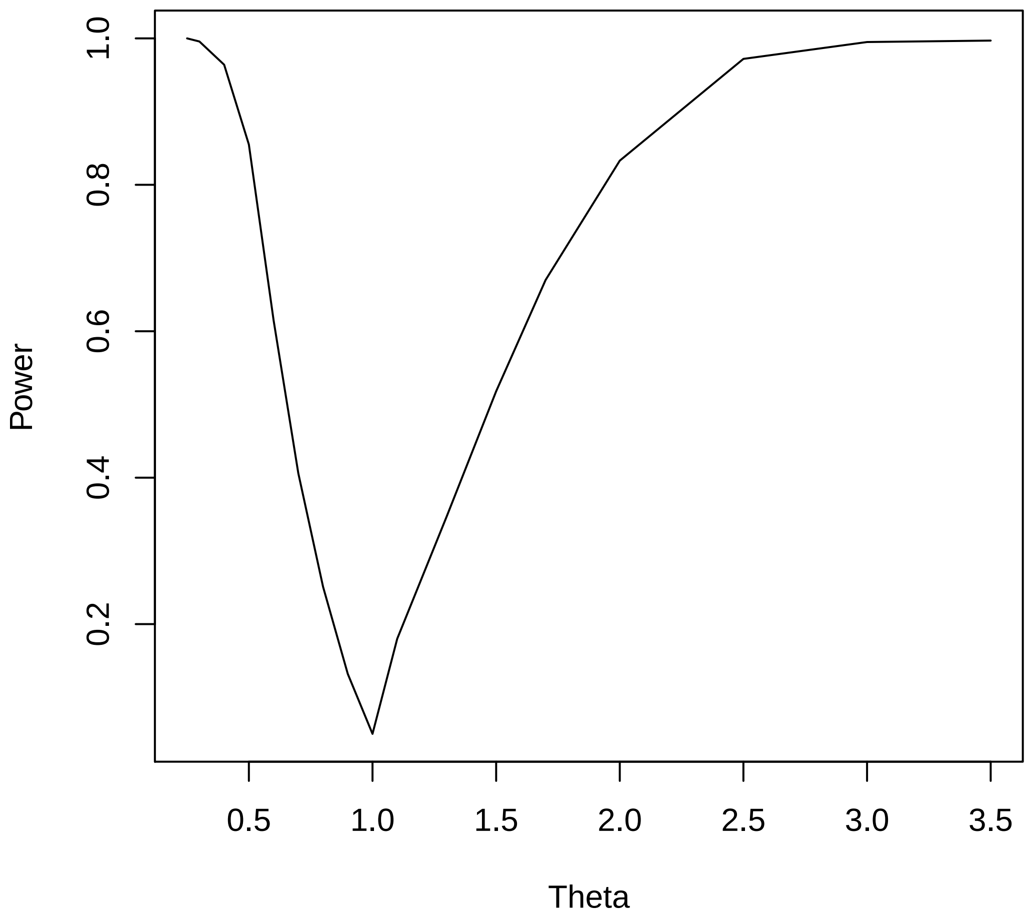

Figure 1: Power curve of the CAT for two sided alternative (with $n = 23$ and $\alpha = 0.05$) plotted against θ

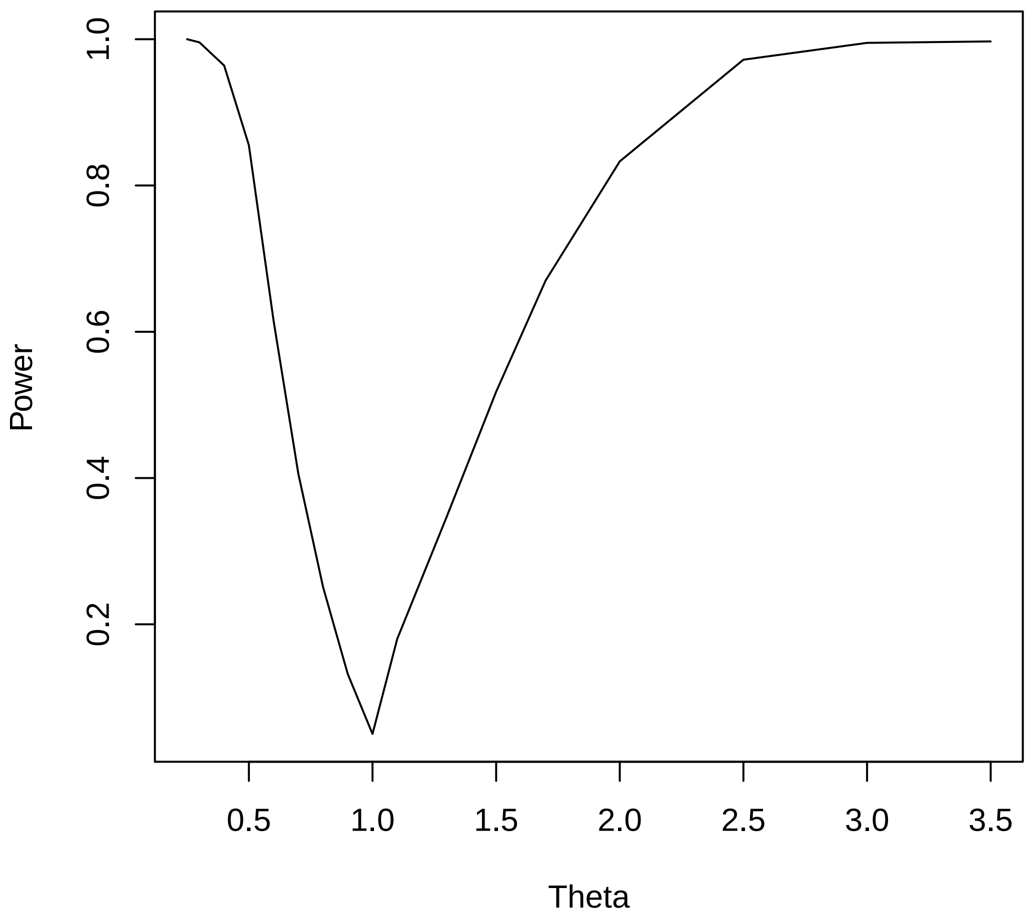

Figure 2: Power curve of the CAT for two sided alternative (with $n = 30$ and $\alpha = 0.05$) plotted against θ